

# $\mathbb{F}_1$ -schemes and toric varieties

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**Abstract:** In this paper it is shown that integral  $\mathbb{F}_1$ -schemes of finite type are essentially the same as toric varieties. A description of the  $\mathbb{F}_1$ -zeta function in terms of toric geometry is given. Etale morphisms and universal coverings are introduced.

## Introduction

There are by now several attempts to make the theory of the field of one element  $\mathbb{F}_1$  rigorous. In [10] the authors formalize the transition from rings to schemes on a categorical level and apply this machinery to the category of sets to obtain the category of  $\mathbb{F}_1$ -schemes as in [1]. In [3] and [5] the authors extend the definition of rings in order to capture a structure that deserves to be called  $\mathbb{F}_1$ . In [1] the author tried instead to fix the minimum properties any of these theories must share. The current paper extends this line of thought. We use terminology of [1] and [2].

In this paper, a ring will always be commutative with unit and a monoid will always be commutative. An *ideal*  $\mathfrak{a}$  of a monoid  $A$  is a subset with  $A\mathfrak{a} \subset \mathfrak{a}$ . A *prime ideal* is an ideal  $\mathfrak{p}$  such that  $S_{\mathfrak{p}} = A \setminus \mathfrak{p}$  is a submonoid of  $A$ . For a prime ideal  $\mathfrak{p}$  let  $A_{\mathfrak{p}} = S_{\mathfrak{p}}^{-1}A$  be the *localization* at  $\mathfrak{p}$ . The *spectrum* of

a monoid  $A$  is the set of all prime ideals with the obvious Zariski-topology (see [1]). Similar to the theory of rings, one defines a structure sheaf  $\mathcal{O}_X$  on  $X = \text{spec}(A)$ , and one defines a *scheme over  $\mathbb{F}_1$*  to be a topological space together with a sheaf of monoids, locally isomorphic to spectra of monoids.

A  $\mathbb{F}_1$ -scheme  $X$  is *of finite type*, if it has a finite covering by affine schemes  $U_i = \text{spec}(A_i)$  such that each  $A_i$  is a finitely generated monoid. For a ring  $R$ , we write  $X_R$  for the  $R$ -base-change of  $X$ , so  $X_R = X_{\mathbb{Z}} \otimes_{\mathbb{Z}} R$ .

For a monoid  $A$  we let  $A \otimes \mathbb{Z}$  be the monoidal ring  $\mathbb{Z}[A]$ . This defines a functor from monoids to rings which is left adjoint to the forgetful functor that sends a ring  $R$  to the multiplicative monoid  $(R, \times)$ . This construction is compatible with gluing, so one gets a functor  $X \mapsto X_{\mathbb{Z}}$  from  $\mathbb{F}_1$ -schemes to  $\mathbb{Z}$ -schemes. In [2] we have shown that  $X$  is of finite type if and only if  $X_{\mathbb{Z}}$  is a  $\mathbb{Z}$ -scheme of finite type.

We say that the monoid  $A$  is *integral*, if it has the cancellation property, i.e., if  $ab = ac$  implies  $b = c$  in  $A$ . This is equivalent to saying that  $A$  injects into its quotient group or  $A$  is a submonoid of a group.

By a *module* of a monoid  $A$  we mean a set  $M$  together with a map  $A \times M \rightarrow M$ ;  $(a, m) \mapsto am$  with  $1m = m$  and  $(ab)m = a(bm)$ . A *stationary point* of a module is a point  $m \in M$  with  $am = m$  for every  $a \in A$ . A *pointed module* is a pair  $(M, m_0)$  consisting of an  $A$ -module  $M$  and a stationary point  $m_0 \in M$ .

## 1 Flatness

Recall the tensor product of two modules  $M, N$  of  $A$ :

$$M \otimes N = M \otimes_A N = M \times N / \sim,$$

where  $\sim$  is the equivalence relation generated by  $(am, n) \sim (m, an)$  for every  $a \in A, m \in M, n \in N$ . The class of  $(m, n)$  is written as  $m \otimes n$ . The tensor product  $M \otimes N$  becomes a module via  $a(m \otimes n) = (am) \otimes n$ . For example, the module  $A \otimes M$  is isomorphic to  $M$ .

Let now  $(M, m_0)$  and  $(N, n_0)$  be two pointed modules of  $A$ , then  $(M \otimes N, m_0 \otimes n_0)$  is a pointed module, called the pointed tensor product.

The category  $\text{Mod}_0(A)$  of pointed modules and pointed morphisms has a

terminal and initial object 0, so it makes sense to speak of kernels and cokernels. It is easy to see that every morphism  $f$  in  $\text{Mod}_0(A)$  possesses both. One defines the *image* of  $f$  as  $\text{im}(f) = \ker(\text{coker}(f))$  and the *coimage* as  $\text{coim}(f) = \text{coker}(\ker(f))$ .

A morphism is called *strong*, if the natural map from  $\text{coim}(f)$  to  $\text{im}(f)$  is an isomorphism. Kernels and cokernels are strong. If  $A \xrightarrow{f} B \xrightarrow{g} C$  is given with  $g$  being strong and  $gf = 0$ , then the induced map  $\text{coker}(f) \rightarrow C$  is strong. Likewise, if  $f$  is strong and  $gf = 0$ , then the induced map  $A \rightarrow \ker g$  is strong. A map is strong if and only if it can be written as a cokernel followed by a kernel.

The usual notion of exact sequences applies, and we say that a sequence of morphisms is *strong exact* if it is exact and all morphisms in the sequence are strong.

A module  $F \in \text{Mod}_0(A)$  is called *flat*, if the functor  $X \mapsto F \otimes X$  is strong-exact, i.e., if for every strong exact sequence

$$0 \rightarrow M \rightarrow N \rightarrow P \rightarrow 0$$

the induced sequence

$$0 \rightarrow F \otimes M \rightarrow F \otimes N \rightarrow F \otimes P \rightarrow 0$$

is strong exact as well.

It is easy to see that a pointed module  $F$  is flat if and only if for every injection  $M \hookrightarrow N$  of pointed modules the map  $F \otimes M \rightarrow F \otimes N$  is an injection.

*Examples.* If  $A$  is a group, then every module is flat. Let  $S$  be a submonoid of  $A$ . Then the localization  $S^{-1}A$  is a flat  $A$ -module. The direct sum  $G \oplus F$  of two flat modules is flat. Finally, consider the free monoid in one generator  $C_+ = \{1, \tau, \tau^2, \dots\}$ , then an  $A$ -module  $M$  is flat if and only if  $\tau m = \tau m'$  implies  $m = m'$  for all  $m, m' \in M$ . This is equivalent to saying that  $M$  is a  $C_+$ -submodule of a module of the quotient group  $C_\infty = \tau^{\mathbb{Z}}$  of  $C_+$ . The same characterization holds for every integral monoid.

A morphism  $\varphi : A \rightarrow B$  of monoids is called flat if  $B$  is flat as an  $A$ -module. A morphism of  $\mathbb{F}_1$ -schemes  $f : X \rightarrow Y$  is called flat if for every  $x \in X$  the morphism of monoids  $f^\# : \mathcal{O}_{Y, f(x)} \rightarrow \mathcal{O}_{X, x}$  is flat.

The following is straightforward.

- A morphism of monoids  $\varphi : A \rightarrow B$  is flat if and only if the induced morphism of  $\mathbb{F}_1$ -schemes  $\text{spec } B \rightarrow \text{spec } A$  is flat.
- The composition of flat morphisms is flat.
- The base change of a flat morphism by an arbitrary morphism is flat.

**Remark.** It is easy to see that if  $\mathbb{Z}[F]$  is flat as  $\mathbb{Z}[A]$ -module, then  $F$  is flat as  $A$ -module. The converse is already false if  $A$  is a group. As an example let  $k$  be a field and let  $A$  be the group of all matrices of the form  $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$  where  $x \in k$ . Let  $A$  act on  $k^2$  in the usual way and trivially on  $k$ . Consider the exact sequence of  $\mathbb{Z}[A]$ -modules,

$$0 \longrightarrow k \xrightarrow{\alpha} k^2 \xrightarrow{\beta} k \longrightarrow 0,$$

where  $\alpha(x) = \begin{pmatrix} x \\ 0 \end{pmatrix}$ ,  $\beta \begin{pmatrix} x \\ y \end{pmatrix} = y$ . Let  $F = \{1\}$  the trivial  $A$ -module, then for every  $\mathbb{Z}[A]$ -module  $M$  one has  $M \otimes_{\mathbb{Z}[A]} \mathbb{Z}[F] = H_0(A, M)$ . Note that  $H_0(A, k) = k$  and that  $H_0(\alpha) = 0$ , so it is not injective, hence  $\mathbb{Z}[F]$  is not flat.

## 2 Algebraic extensions

Let  $A$  be a submonoid of  $B$ . An element  $b \in B$  is called *algebraic over  $A$* , if there exists  $n \in \mathbb{N}$  with  $b^n \in A$ . The extension  $B/A$  is called *algebraic*, if every  $b \in B$  is algebraic over  $A$ . An algebraic extension  $B/A$  is called *strictly algebraic*, if for every  $a \in A$  the equation  $x^n = a$  has at most  $n$  solutions in  $B$ .

If  $B/A$  is algebraic, then  $\mathbb{Z}[B]/\mathbb{Z}[A]$  is an algebraic ring extension, but the converse is wrong in general, as the following example shows: Let  $A = \mathbb{F}_1$  and  $B$  be the set of two elements, 1 and  $b$  with  $b^2 = b$ .

A monoid  $A$  is called *algebraically closed*, if every equation of the form  $x^n = a$  with  $a \in A$  has a solution in  $A$ . Every monoid  $A$  can be embedded into

an algebraically closed one, and if  $A$  is a group, then there exists a smallest such embedding, called the *algebraic closure* of  $A$ . For example, the algebraic closure  $\bar{\mathbb{F}}_1$  of  $\mathbb{F}_1$  is the group  $\mu_\infty$  of all roots of unity, which is isomorphic to  $\mathbb{Q}/\mathbb{Z}$ .

### 3 Etale morphisms

Recall that a homomorphism  $\varphi: A \rightarrow B$  of monoids is called a *local* homomorphism, if  $\varphi^{-1}(B^\times) = A^\times$  (every  $\varphi$  satisfies “ $\supset$ ”). For a monoid  $A$  let  $m_A = A \setminus A^\times$  be its maximal ideal. It is easy to see that a homomorphism  $\varphi: A \rightarrow B$  is local if and only if  $\varphi(m_A) \subset m_B$ .

A local homomorphism  $\varphi: A \rightarrow B$  is called *unramified* if

- $\varphi(m_A)B = m_B$  and
- $\varphi$  injects  $A^\times$  into  $B^\times$  and  $B/\varphi(A)$  is a finite strictly algebraic extension.

Note that if  $\varphi$  is unramified, then so are all localizations  $\varphi_{\mathfrak{p}}: A_{\varphi^{-1}(\mathfrak{p})} \rightarrow B_{\mathfrak{p}}$  for  $\mathfrak{p} \in \text{spec } B$ .

A morphism  $f: X \rightarrow Y$  of  $\mathbb{F}_1$ -schemes is called unramified, if for every  $x \in X$  the local morphism  $f^\# : \mathcal{O}_{Y,f(x)} \rightarrow \mathcal{O}_{X,x}$  is unramified.

A morphism  $f: X \rightarrow Y$  of  $\mathbb{F}_1$ -schemes is called *locally of finite type*, if every point in  $Y$  has an open affine neighborhood  $V = \text{spec } A$  such that  $f^{-1}(V)$  is a union of open affines  $\text{spec } B_i$  with  $B_i$  finitely generated as a monoid over  $A$ . The morphism is *of finite type* if for every point in  $Y$  the number of  $B_i$  can be chosen finite. The morphism is called *finite*, if every  $y \in Y$  has an open affine neighborhood  $V = \text{spec } A$  such that  $f^{-1}(V)$  is affine, equal to  $\text{spec } B$ , where  $B$  is finitely generated as  $A$ -module.

A morphism  $f: X \rightarrow Y$  of finite type is called *étale*, if  $f$  is flat and unramified. It is called an *étale covering*, if it is also finite.

**Proposition 3.1** *The étale coverings of  $\text{spec } \mathbb{F}_1$  are the morphisms of the form  $\text{spec } A \rightarrow \text{spec } \mathbb{F}_1$ , where  $A$  is a finite cyclic group. The scheme  $\text{spec } \bar{\mathbb{F}}_1$  has no non-trivial étale coverings.*

**Proof:** Clear. □

A connected scheme over  $\mathbb{F}_1$ , which has only the trivial étale covering, is called *simply connected*.

**Proposition 3.2** *The schemes  $\operatorname{spec} \bar{\mathbb{F}}_1$ ,  $\operatorname{spec} C_+ \times_{\mathbb{F}_1} \bar{\mathbb{F}}_1$  and  $\mathbb{P}_{\bar{\mathbb{F}}_1}^1$  are simply connected.*

**Proof:** The first has been dealt with. For the second, let  $A = \mu_\infty \times C_+$ . Then  $\operatorname{spec} A = \operatorname{spec} C_+ \times_{\mathbb{F}_1} \bar{\mathbb{F}}_1$ . Let  $f : X \rightarrow \operatorname{spec} A$  be an étale covering. As  $f$  is finite,  $X$  is affine, say  $X = \operatorname{spec} B$ . Let  $\varphi : A \rightarrow B$  denote the corresponding morphism of monoids. The space  $\operatorname{spec} A$  consists of two points, the generic point  $\eta_A$  and the closed point  $c_A$ . Likewise, let  $\eta_B, c_B$  denote the generic and closed points of  $\operatorname{spec} B$ . One has  $f(\eta_B) = \eta_A$ . We will show that  $f(c_B) = c_A$ . Assume the contrary. Then  $\varphi^{-1}(m_B)$  is empty, hence  $\varphi$  maps  $A$  to the unit group  $B^\times$ . The localization at the closed point  $c_B$  then maps  $\mu_\infty \times C_\infty$  to  $B^\times$  and is unramified, hence injective. But as  $C_+ \rightarrow C_\infty$  is not finite, neither can  $\varphi$  be finite, a contradiction. So we conclude  $f(c_B) = c_A$ , and so the corresponding localization, which is  $\varphi$  itself, is unramified. Let  $s = \varphi(\tau)$ , where  $\tau$  is the generator of  $C_+$ . Then  $\varphi(m_A)B = m_B$  implies  $m_B = sB$ , and so  $B = B^\times \cup sB$  (disjoint union). Also,  $B^\times$  is an algebraic extension of  $A^\times \cong \mu_\infty$ , hence equals  $\varphi(A^\times)$ . As  $B$  is finitely generated and flat as  $A$ -module, there are  $b_1, \dots, b_r \in B$  with

$$sB = B^\times s^{\mathbb{N}} \cup B^\times s^{\mathbb{N}} b_1 \cup \dots \cup B^\times s^{\mathbb{N}} b_r.$$

If we assume  $r > 0$ , then  $b_1$  is algebraic over  $\varphi(A) = B^\times \cup B^\times s^{\mathbb{N}}$ , so let  $N$  be the smallest number in  $\mathbb{N}$  such that  $b_1^N \in \varphi(A)$ . Then  $b_1^N \notin B^\times \cong \mu_\infty$ , because, as the extension is strictly algebraic, then  $b_1$  would be in  $B^\times$  already. So  $b_1^N \in B^\times s^{\mathbb{N}}$ . As the group  $B^\times$  is divisible, we can replace  $b_1$  with a  $B^\times$  multiple to get  $b_1^N = s^M$  for some  $M \in \mathbb{N}$ . Then  $b_1 \notin B^\times s^{\mathbb{N}}$ , as  $b_1 = b^* s^k b_1$  leads to  $s^M = b_1^N = (b^*)^N s^{kN+M}$  which contradicts the injectivity of  $\varphi$ . But then  $b_1$  must be in one of the other  $B^\times s^{\mathbb{N}}$ -orbits, which contradicts the disjointness of these orbits. We conclude  $r = 0$ , i.e.  $B = B^\times \cup B^\times s^{\mathbb{N}} \cong A$  as claimed. The assertion for  $\mathbb{P}_{\bar{\mathbb{F}}_1}^1$  is an easy consequence. □

## 4 Toric varieties

Recall a *toric variety* is an irreducible variety  $V$  over  $\mathbb{C}$  together with an algebraic action of the  $r$ -dimensional torus  $\mathrm{GL}_1^r$ , such that  $V$  contains an open orbit.

As toric varieties can be constructed via lattices it follows that every toric variety is the lift  $X_{\mathbb{C}}$  of an  $\mathbb{F}_1$ -scheme  $X$ . For integral schemes of finite type there is a converse direction given in the following theorem, which shows that integral  $\mathbb{F}_1$ -schemes of finite type are essentially the same as toric varieties.

**Theorem 4.1** *Let  $X$  be a connected integral  $\mathbb{F}_1$ -scheme of finite type. Then every irreducible component of  $X_{\mathbb{C}}$  is a toric variety. The components of  $X_{\mathbb{C}}$  are mutually isomorphic as toric varieties.*

**Proof:** Let  $U = \mathrm{spec} A$  be an open affine subset of  $X$ . Let  $\eta$  be the generic point of  $X$ , then the localization  $G = A_{\eta}$  is the quotient group of  $A$ . At the same time,  $G$  is the stalk  $\mathcal{O}_{X,\eta}$ , so  $G$  does not depend on the choice of  $U$  up to canonical isomorphism. Let  $\varphi : A \rightarrow G$  be the quotient map, which is injective as  $X$  is integral. The  $\mathbb{C}$ -algebra homomorphism,

$$\begin{aligned} \mathbb{C}[A] &\rightarrow \mathbb{C}[G] \otimes \mathbb{C}[A] \\ a &\mapsto \varphi(a) \otimes a \end{aligned}$$

defines an action of the algebraic group  $\mathcal{G} = \mathrm{spec} \mathbb{C}[G]$  on  $\mathrm{spec} \mathbb{C}[A]$ . Since this is compatible with the restriction maps of the structure sheaf, we get an algebraic action of the group scheme  $\mathcal{G}$  on  $X_{\mathbb{C}}$ . As  $X$  is integral,  $\mathcal{G} = \mathrm{spec} \mathbb{C}[G] = \mathrm{spec} \mathbb{C}[A_{\eta}]$  also is an open subset  $V_{\mathbb{C}}$  of  $X_{\mathbb{C}}$ , and for  $U_{\mathbb{C}} = \mathrm{spec} \mathbb{C}[A]$  the map

$$\mathcal{O}(U_{\mathbb{C}}) = \mathbb{C}[A] \xrightarrow{\varphi} \mathbb{C}[G] = \mathcal{O}(V_{\mathbb{C}})$$

is the restriction map of the structure sheaf  $\mathcal{O}$  of  $X_{\mathbb{C}}$ . The map  $\mathbb{C}[A] \rightarrow \mathbb{C}[G]$  is injective and  $\mathbb{C}[G]$  has zero Jacobson radical, so it follows that  $V_{\mathbb{C}}$  is dense in  $X_{\mathbb{C}}$ , so in particular it meets every irreducible component. The group  $G$  is a finitely generated abelian group, so  $G \cong \mathbb{Z}^r \times F$  for a finite abelian group

$F$ . Hence  $\mathcal{G} \equiv \mathrm{GL}_1^r \times F$  as a group-scheme. As  $\mathcal{G}$  meets every component of  $X_{\mathbb{C}}$ , the latter are permuted by  $F$ . Whence the claim.  $\square$

To formulate the next result, we will briefly recall the standard construction of toric varieties, see [4]. Let  $N$  be a *lattice*, i.e., a group isomorphic to  $\mathbb{Z}^n$  for some  $n$ . A *fan*  $\Delta$  in  $N$  is a finite collection of *proper convex rational polyhedral cones*  $\sigma$  in the real vector space  $N_{\mathbb{R}} = N \otimes \mathbb{R}$ , such that every face of a cone in  $\Delta$  is in  $\Delta$  and the intersection of two cones in  $\Delta$  is a face of each. (Here zero is considered a face of every cone.) We explain the notation further: A *convex cone* is a convex subset  $\sigma$  of  $N_{\mathbb{R}}$  with  $\mathbb{R}_{\geq 0}\sigma = \sigma$ , it is *polyhedral*, if it is finitely generated and *rational*, if the generators lie in the lattice  $N$ . Finally, a cone is called *proper* if it does not contain a non-zero sub vector space of  $N_{\mathbb{R}}$ .

Let a fan  $\Delta$  be given. Let  $M = \mathrm{Hom}(N, \mathbb{Z})$  be the dual lattice. for a cone  $\sigma \in \Delta$  the *dual cone*  $\check{\sigma}$  is the cone in the dual space  $M_{\mathbb{R}}$  consisting of all  $\alpha \in M_{\mathbb{R}}$  such that  $\alpha(\sigma) \geq 0$ . This defines a monoid  $A_{\sigma} = \check{\sigma} \cap M$ . Set  $U_{\sigma} = \mathrm{spec}(\mathbb{C}[A_{\sigma}])$ . If  $\tau$  is a face of  $\sigma$ , then  $A_{\tau} \supset A_{\sigma}$ , and this inclusion gives rise to an open embedding  $U_{\tau} \hookrightarrow U_{\sigma}$ . Along these embeddings we glue the affine varieties  $U_{\sigma}$  to obtain a variety  $X_{\Delta}$  over  $\mathbb{C}$ , which has a given  $\mathbb{F}_1$ -structure. Then  $X_{\Delta}$  is a toric variety, the torus being  $U_0 \cong \mathrm{GL}_1^n$ . Every toric variety is given in this way.

**Lemma 4.2** *Let  $B$  be a submonoid of the monoid  $A$  of finite index. Then the map  $\psi : \mathrm{spec} A \rightarrow \mathrm{spec} B$  defined by  $\psi(\mathfrak{p}) = \mathfrak{p} \cap B$  is a bijection.*

**Proof:** Let  $N \in \mathbb{N}$  be such that  $a^N \in B$  for every  $a \in A$ . To see injectivity, let  $\psi(\mathfrak{p}) = \psi(\mathfrak{q})$  and let  $a \in \mathfrak{p}$ . Then  $a^N \in \mathfrak{q}$  and so  $a \in \mathfrak{q}$  as  $\mathfrak{q}$  is a prime ideal. This shows  $\mathfrak{p} \subset \mathfrak{q}$  and by symmetry we get equality. For surjectivity, let  $\mathfrak{p}_B \in \mathrm{spec} B$  and let  $\mathfrak{p} = \{a \in A : a^N \in \mathfrak{p}_B\}$ . Then  $\psi(\mathfrak{p}) = \mathfrak{p}_B$ .  $\square$

**Proposition 4.3** *Suppose that  $\Delta$  is a fan in a lattice of dimension  $n$ . For  $j = 0, \dots, n$  let  $f_j$  be the number of cones in  $\Delta$  of dimension  $j$ . Set*

$$c_j = \sum_{k=j}^n f_{n-k} (-1)^{k+j} \binom{k}{j}.$$



Let  $X$  be the corresponding toric variety, then the  $\mathbb{F}_1$ -zeta function of  $X$  equals

$$\zeta_X(s) = s^{c_0}(s-1)^{c_1} \cdots (s-n)^{c_n}.$$

**Proof:** Let  $\sigma \in \Delta$  be a cone of dimension  $k$ . Let  $F$  be a face of  $\check{\sigma}$ . Let  $\mathfrak{p}_F = A_\sigma \setminus F$ . Then  $\mathfrak{p}_F$  is a non-empty prime ideal in  $A_\sigma$ . The map  $F \mapsto \mathfrak{p}_F$  is a bijection between the set of all faces of  $\check{\sigma}$  and the set of non-empty prime ideals of  $A_\sigma$ . The set  $S_{\mathfrak{p}} = A \setminus \mathfrak{p}$  equals  $M \cap F$ . The quotient group  $\text{Quot}(S_{\mathfrak{p}})$  is isomorphic to  $\mathbb{Z}^f$ , where  $f$  is the dimension of  $F$ . There is a bijection between the set of faces of  $\sigma$  and the set of faces of  $\check{\sigma}$  mapping a face  $\tau$  to the face  $F$  of all  $\alpha \in \check{\sigma}$  with  $\alpha(\tau) = 0$ . The dimension of  $F$  then equals  $n - \dim(\tau)$ . So let  $f_j^\sigma$  denote the number of faces of  $\sigma$  of dimension  $j$ . Then the zeta polynomial of  $X_\sigma$  equals

$$N_\sigma(x) = \sum_{k=0}^n f_k^\sigma (x-1)^{n-k}.$$

Let  $N_\Delta$  be the zeta polynomial of  $X_\Delta$ . We get

$$\begin{aligned} N_\Delta(x) &= \sum_{k=0}^n f_k (x-1)^{n-k} \\ &= \sum_{k=0}^n f_k \sum_{j=0}^{n-k} \binom{n-k}{j} x^j (-1)^{n-k-j} \\ &= \sum_{k=0}^n f_{n-k} \sum_{j=0}^k \binom{k}{j} x^j (-1)^{k-j} \\ &= \sum_{j=0}^n x^j \sum_{k=j}^n f_{n-k} \binom{k}{j} (-1)^{k-j}. \end{aligned}$$

This implies the claim. □

## 5 Valuations

On the infinite cyclic monoid  $C_+ = \{1, \tau, \tau^2, \dots\}$  we have a natural linear order given by  $\tau^k \leq \tau^l \Leftrightarrow k \leq l$ . Let  $\varphi, \psi$  be two monoid morphisms from a

monoid  $A$  to  $C_+$ . Then define  $\varphi \leq \psi \Leftrightarrow \varphi(a) \leq \psi(a) \forall a \in A$ . A *valuation* on  $A$  is a non-trivial homomorphism  $v : A \rightarrow C_+$  which is minimal with respect to the order  $\leq$  among all non-trivial homomorphisms from  $A$  to  $C_+$ . Let  $V(A)$  denote the set of valuations on  $A$ .

**Lemma 5.1** *Let*

$$1 \longrightarrow A \longrightarrow B \xrightarrow{\varphi} F \longrightarrow 1$$

*be an exact sequence of monoids, where  $F$  is a finite abelian group. Then for every valuation  $v \in V(A)$  there exists a unique valuation  $w$  on  $B$  and  $k \in \mathbb{N}$  such that*

$$w|_A = v^k.$$

*Mapping  $v$  to  $w$  sets up a bijection from  $V(A)$  to  $V(B)$ .*

**Proof:** Let  $F'$  be a subgroup of  $F$  and let  $B'$  be the preimage of  $F'$  under  $\varphi$ . We get two exact sequences

$$1 \longrightarrow A \longrightarrow B' \longrightarrow F' \longrightarrow 1,$$

and

$$1 \longrightarrow B' \longrightarrow B \longrightarrow F/F' \longrightarrow 1.$$

Assume we have proven the lemma for each of these two sequences, then it follows for the original one. In this way we reduce the proof to the case when  $F$  is a finite cyclic group. We first show existence of  $w$  for given  $v$ . For this let  $f_0$  be a generator of  $F$  and let  $l$  be its order. Choose a  $b_0$  in the preimage  $\varphi^{-1}(f_0)$ . Then  $b_0^l \in A$ , and  $v(b_0^l) = \tau^n$  for some  $n \geq 0$ . If  $n = 0$ , then set  $k = 1$  and define  $w : B \rightarrow C_+$  by  $w(b_0^j a) = v(a)$  for  $a \in A$  and  $j \geq 0$ . If  $n > 0$ , then set  $k = l/\gcd(l, n)$  and let  $w : B \rightarrow C_+$  be defined by  $w(b_0^j a) = \tau^j v(a)^k$ . This shows existence of the extension  $w$ .  $\square$

## 6 Cohomology

Cohomology is not defined over  $\mathbb{F}_1$ . I am grateful to Ofer Gabber for bringing the following example to my attention. Let  $X$  be the topological space consisting of three points  $\eta, X_+, x_-$ . The open sets besides the trivial ones are

$U = \{\eta\}, U_+ = \{\eta, x_+\}, U_- = \{\eta, x_-\}$ . Let  $A$  be a subgroup of the abelian group  $B$  and let  $C = B/A$ . Let  $\mathcal{F}$  be the sheaf of abelian groups on  $X$  with  $\mathcal{F}(U_\pm) = A$  and  $\mathcal{F}(U) = B$  and the restriction being the inclusion. Let  $\mathcal{G}$  be the constant sheaf  $B$  and let  $\mathcal{H}$  be the quotient sheaf  $\mathcal{G}/\mathcal{F}$ . As  $\mathcal{G}$  is flabby, the long cohomology sequence terminates and looks like this:

$$0 \rightarrow H^0(\mathcal{F}) \rightarrow H^0(\mathcal{G}) \rightarrow H^0(\mathcal{H}) \rightarrow H^1(\mathcal{F}) \rightarrow 0$$

In concrete terms this is

$$0 \rightarrow A \rightarrow B \rightarrow C \times C \rightarrow (C \times C)/\Delta \rightarrow 0,$$

where  $\Delta$  means the diagonal in  $C \times C$ . Let  $f : X \rightarrow X$  be the homeomorphism with  $f(x_+) = x_-$ ,  $f(x_-) = x_+$ , and  $f(\eta) = \eta$ . There is a natural isomorphism  $f_*\mathcal{F} \cong \mathcal{F}$  and for the other sheaves as well. On the global sections of  $\mathcal{F}$  and  $\mathcal{G}$  this induces the trivial map, whereas on  $H^0(\mathcal{H})$  it induces the flip  $(a, b) \mapsto (b, a)$ , which on  $H^1(\mathcal{F})$  amounts to the same as the inversion  $a \mapsto -a$ . The naturality of these isomorphisms means that if the sheaves and the cohomology groups are defined over  $\mathbb{F}_1$ , then so must be the flip. This, however, is not the case, as for a set  $S$  the inversion on the abelian group  $\mathbb{Z}[S]$  is not induced by a self-map of  $S$ .

Even more convincing is the fact that in this example there are different injective resolutions which produce different cohomology groups.

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